

# Optimal recovery of solutions to Dirichlet Problems for Laplace's Equation

Najia Bacha, Dr.Yuliya Babenko (advisor)

KSU

April 16, 2015

# Laplace's equation

- The following equation is called Laplace's equation

$$\Delta U = 0.$$

Which is a partial differential equation of second order.

- The Laplacian of  $U$  is  $\Delta U = \sum_{i=1}^n U_{x_i x_i}$
- The solution  $U$  is a  $C^2$  harmonic function.

$$U : \bar{\Omega} \rightarrow \mathbb{R}$$

$$X \mapsto U(x)$$

Where  $\Omega$  is a given open set,  $\bar{\Omega} \subset \mathbb{R}^n$

# Applications of Laplace's Equation

- Steady-State Heat Conduction.
- Statics Deflection of a Membrane.
- Electrostatic Potential.

# Dirichlet's problem

When we add the boundary condition to Laplace's equation, we obtain Dirichlet's problem:

$$\Delta U = 0 \text{ in } \Omega$$

$$U = f \text{ on } \partial\Omega$$

The solution to this problem is well known. For every element  $\mathbf{x}$  of the domain we have

$$U(\mathbf{x}) = - \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y}$$

Where  $f$  is the given function,  $\frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}}$  is the normal derivative of the Green function.

# Questions

**Question:** what if the function  $f$  is not fully known?

- What if  $f$  is known at  $n$  fixed points?
- What if  $f$  is known at  $n$  points, but with error ?
- What if  $f$  is known at  $n$  points, but we have a choice where to take those measurements without error?
- What if  $f$  is known at  $n$  first Fourier coefficients?

**This project:** we assumed that  $f$  is known at  $n$  points on the boundary  $\partial\Omega$ :

$$f(x_1), f(x_2), \dots, f(x_n) \text{ and } f \in H^\omega(\partial\Omega)$$

Where  $H^\omega$  is a certain class of smoothness defined with the help of  $\omega$ .

**Goal:** develop an optimal method of recovery of solution to the Dirichlet problem based on incomplete information about  $f$  and to compute the (optimal) error.

# Definitions

## Definition

*Method of recovery is an operator such that*

$$\Phi : \mathbb{R}^n \rightarrow C^2(\Omega) \cap H^\omega(\partial\Omega), \quad \bar{y} \mapsto \Phi(\bar{y})$$

## Definition

*Error of recovery by method  $\Phi$  based on the given information  $\bar{y} = (f(x_1), f(x_2), \dots, f(x_n))$  is defined in  $\partial\Omega$ .*

$$E(\Phi, \bar{y}) = \sup_{f \in H^\omega} \|U(\mathbf{x}) - \Phi(\bar{y})\|$$

*Where  $\Phi$  is the method of recovery,  $\bar{y}$  is the given data point, and  $U$  is the actual solution.*

## Definition

*The optimal error*

$$E(\bar{y}) = \inf_{\Phi} E(\Phi, \bar{y})$$

# Main results

## 1 Construct an Extremal Function

$$f_\rho(\mathbf{x}) = \min\{\omega(|\mathbf{x} - \mathbf{x}_i|)\}, \quad 1 \leq i \leq n$$

In every cell

$$\Pi_i = \{\mathbf{x} : f_\rho(\mathbf{x}) = \omega(\rho(\mathbf{x}, \mathbf{x}_i))\}, \quad 1 \leq i \leq n$$

$$C_f(\mathbf{y}) = y_i \text{ On } \Pi_i$$

## 2 Method of Recovery

$$\Phi^* : \tilde{U}(\mathbf{x}) \simeq - \int_{\partial\Omega} C_f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y}$$

Where  $C_f$  is an approximant to  $f$

### Theorem

*The optimal error*

$$\inf_{\Phi} \sup_{f \in H^\omega} \|U(\mathbf{x}) - \Phi(\bar{y})\|_{L_1} = \|U(\mathbf{x}) - \tilde{U}(\mathbf{x})\|_{L_1} = \int_{\partial\Omega} f_\rho(\mathbf{y}) \left( \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{x} \right) d\mathbf{y}$$

## Proof: Estimate from Above Part 1

$$\begin{aligned}\|U(\mathbf{x}) - \tilde{U}(\mathbf{x})\|_{L_1} &= \int_{\Omega} |U(\mathbf{x}) - \tilde{U}(\mathbf{x})| d\mathbf{x} \\ &= \int_{\Omega} \left| - \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} + \int_{\partial\Omega} C_f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} \right| d\mathbf{x} \\ &= \int_{\Omega} \left| \int_{\partial\Omega} (C_f(\mathbf{y}) - f(\mathbf{y})) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} \right| d\mathbf{x}\end{aligned}$$

We have

$$\int_{\partial\Omega} (C_f(\mathbf{y}) - f(\mathbf{y})) d\mathbf{y} = \sum_{i=1}^n \int_{\Pi_i} (y_i - f(\mathbf{y})) d\mathbf{y}$$

Estimate on each  $\Pi_i$ ,  $1 \leq i \leq n$

$$|f(\mathbf{y}) - y_i| = |f(\mathbf{y}) - f(x_i)|$$

On every cell  $\Pi_i$ , we have

$$|f(\mathbf{y}) - f(x_i)| \leq \omega(\rho(\mathbf{y}, x_i)) = f_{\rho}(\mathbf{y})$$



## Proof: Estimate from Above Part2

Therefore on every cell we have  $\Pi_i : |f(\mathbf{y}) - y_i| \leq f_\rho(\mathbf{y})$

$$\begin{aligned}\|U(\mathbf{x}) - \tilde{U}(\mathbf{x})\|_{L_1} &\leq \int_{\Omega} \left| \int_{\partial\Omega} f_\rho(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} \right| d\mathbf{x} \\ &\leq \int_{\Omega} f_\rho(\mathbf{y}) \left( \int_{\partial\Omega} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{x} \right) d\mathbf{y}\end{aligned}$$

# Proof: Estimate from Below

Estimate from below

$$\sup_{f \in H^\omega} \left\| - \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} - \Phi(\bar{\mathbf{y}}) \right\| \geq \sup_{f \in H^\omega} \left\| - \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} - \Phi(0) \right\|$$

$$= \sup_{f \in H^\omega} \max \left( \left\| - \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} - \Phi(0) \right\|, \left\| \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} - \Phi(0) \right\| \right)$$

Using the fact that  $\max(\|a\|, \|b\|) \geq \frac{1}{2}(\|a\| + \|b\|)$

$$\geq \sup_{f \in H^\omega} \frac{1}{2} \left( \left\| - \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} - \Phi(0) \right\| + \left\| \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} - \Phi(0) \right\| \right)$$

Using the fact that  $(\|a\| + \|b\|) \geq \|a - b\|$

$$\begin{aligned} &\geq \sup_{f \in H^\omega} \frac{1}{2} \left\| - 2 \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} - \Phi(0) + \Phi(0) \right\| \\ &\geq \sup_{f \in H^\omega} \left\| \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} \right\| \\ &\geq \left\| \int_{\partial\Omega} f_\rho(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} \right\| \end{aligned}$$